A NEW COMPLEX VECTOR METHOD FOR
BALANCING CHEMICAL EQUATIONS

NOVA KOMPLEKSNA METODA ZA URAVNOTEŽENJE KEMIČNIH ENAČB

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»Just as tensor analysis is the proper language of kinematics, so linear algebra is the proper language of stoichiometry.«

1 INTRODUCTION

Balancing chemical equations is a basic matter of chemistry, if not one of its most important issues, and it plays a main role in its foundation. Indeed, it is a subtle question which deserves considerable attention.

Also, this topic was a magnet for a number of researchers around the world, because the general problem of balancing chemical equations was considered as one of the hardest problems in chemistry, as well as in mathematics.

In chemical literature there are a great number of papers which consider the problem of balancing chemical equations in different chemical ways, but all of them offer only some particular procedures for balancing simple chemical equations. Very often, these chemical procedures generate absurd results, because most of them are founded on presumed chemical principles, but not on genuine exact principles. And these presumed chemical principles generate paradoxes! The balancing chemical equations does not depend on chemical principles, it is a mathematical operation which is founded on true mathematical principles.

There is a number of paradoxes in chemistry about balancing chemical equations, and these will be systematized and studied in a special article of higher level by the same author which will appear in the near future. In this article only one paradox in balancing chemical equations, discovered in1 is considered.

The author would like to emphasize very clearly, that balancing chemical equations is not chemistry; it is just linear algebra. Although it is true, that it is not chemistry, it is very important for chemistry! In this particular case the following question comes up: If balancing chemical equations is not chemistry, then why is it considered in chemistry? Or maybe more important is this question: If it is linear algebra, then it should be studied in mathematics, so why bother chemists?

The author will address the above questions assuming that: the problem of balancing chemical equations was a multidisciplinary subject and for its solution both mathematicians and chemists are needed. The job of chemists is to perform reactions, while balancing their equations is a job for mathematicians. Mathematics, as a servant to other sciences, is a problem solver, but chemistry is a result user.

The skepticism about balancing chemical equations by chemical principles appeared a long time ago. Let’s quote the opinions of three chemists.

In 1926 the American chemist Simons2 wrote: The balancing of an equation is a mathematical process and independent of chemistry. The order of steps in the process is as essential as the order of steps in long division and the process is much simpler than is ordinary assumed.

Seventy-one years later, the Dutch chemist Ten Hoor3, made a similar statement: Balancing the equation of the reaction is a matter of mathematics only.

Now is the right place to paraphrase the criticism of the American chemist Herndon4: The major changes that have taken place over seven decades are substitutions of the terms ‘change in oxidation number’ and ‘algebraic...
method’ for the terms ‘valence change’ and ‘method of undetermined coefficients’, respectively. This assertion shows very clearly the picture of chemists’ contributions to balancing chemical equations by chemical principles. Probably expressing his satisfaction with the chemists’ contributions to balancing chemical equations, Herndon, the editor of the Journal of Chemical Education, decided that further discussion of equation balancing will not appear in the Journal unless it adds something substantially new to what has already appeared.

In view of the above assertions this question arises: 

Are there in chemistry ‘chemical principles’ capable of offering solutions of the general problem of balancing chemical equations? Maybe more interesting is the question: What are ‘chemical principles’ – a rhetorical sophism of chemists or their hopes?

More interesting for us is to give an answer to both questions from a scientific view point. Perhaps, the more appropriate short answer to the first question is: ‘Chemical principles’ are not defined entities in chemistry, and so this term does not have any meaning. They are not capable to provide solutions of the general problem of balancing chemical equations, because they are founded on an intuitive basis and they represent only a main generator for paradoxes. However, the necessary and sufficient conditions for a complete solution of the general problem of balancing chemical equations lie outside of chemistry, and we must look for them in an amalgamated theory of n-dimensional vector spaces, linear algebra, abstract algebra and topology. It is a very hard problem of the highest level in chemistry and mathematics, which must be considered only on a scientific basis. Like this should look the answer to the first question.

The answer to the second question will be described in this way: Actually, ‘chemical principles’ are a remnant of an old traditional approach in chemistry, when chemists were busy with the verification of their results obtained in chemical experiments. It is true, that until second half of 20th century there was no mathematical method for balancing chemical equations in chemistry, other than Bottomley’s algebraic method. Chemists balanced simple particular chemical equations using only Johnson’s change in oxidation number procedure, Simons – Waldbauer – Thran’s partial reactions procedure and other slightly different modifications derived from them. The ‘chemical principles’ were an assumption of traditional chemists, who thought before the appearance of Jones’ problem, that the solution of the general problem of balancing chemical equations is possible by use of chemical procedures. But, practice showed that the solution of the century old problem is possible only by using contemporary sophisticated mathematical methods.

These questions require a deeper elaboration than given here, and it will be a main concern of the author in his future research.

2 PRELIMINARIES

The exact statements about balancing chemical equations agree with the following well-known results.

**Theorem 1.** Every chemical reaction can be reduced in a matrix equation $Ax = 0$, where $A$ is a reaction matrix, $x$ is a column-vector of the unknown coefficients and $0$ is a null column-vector.

**Proof.** The proof of this theorem immediately follows from $2^9$.

**Remark 2.** The coefficients satisfy three basic principles (corresponding to a closed input-output static model$10,11$)

- the law of conservation of atoms,
- the law of conservation of mass, and
- the time-independence of the reaction.

**Theorem 3.** Every chemical reaction can be presented as a matrix Diophantine equation $Ax = By$, where $A$ and $B$ are matrices of reactants and products, respectively, and $x$ and $y$ are column-vectors of unknown coefficients. The proof of this theorem is given in $12,13$.

This theorem is actually the Jones’ problem founded by virtue of Crocker’s procedure for balancing chemical equations$14$, and from this problem the formalization of chemistry began. This problem is a milestone in chemistry and mathematics as well, and just it opened the door in chemistry to enter a new mathematical freshness. Jones with his problem transferred the general problem of balancing chemical equations from the field of chemistry into the field of mathematics and opened way to solve this problem with mathematical methods founded on principles of linear algebra.

Before the appearance of this problem, the approach of balancing chemical equations was intuitive and useful only for some elementary chemical equations. Now, some well-known results from the theory of complex n-dimensional vector spaces$15,16$, for resolution of balancing chemical equations will be introduced. Here, by $C$ is denoted the set of complex numbers and by $C^n$ is denoted the Euclidian n-dimensional vector space with complex entries.

**Definition 4.** A vector space over the field $C$ consists of a set $V$ of objects called vectors for which the axioms for vector addition hold

- (A1) If $u, v \in V$, then $(u + v) \in V$,
- (A2) $u + v = v + u$, $\forall u, v \in V$,
- (A3) $u + (v + w) = (u + v) + w$, $\forall u, v, w \in V$,
- (A4) $u + 0 = u$, $\forall u \in V$,
- (A5) $-u + u = 0$, $\forall u \in V$,

and the axioms for scalar multiplication

- (S1) If $u \in V$, then $au \in V$, $\forall a \in C$,
- (S2) $(a + b)u = au + bu$, $\forall u, v \in V \land \forall a \in C$,
- (S3) $a(bu) = (ab)u$, $\forall u \in V \land \forall a, b \in C$,
- (S4) $1u = u$, $\forall u \in V$.
The content of axioms (Aₗ) and (Sᵢ) is described with the assertion that V is closed under vector addition and scalar multiplication. The element 0 in axiom Aₗ is called the zero vector.

Definition 6. If V is a vector space over the field C, a subset U of V is called a subspace of V if U itself is a vector space over C, where U uses the vector addition and scalar multiplication of V.

Definition 7. Let V be a vector space over the field C, and let vᵢ ∈ V (1 ≤ i ≤ n). Any vector v in V of the form
\[ v = a₁v₁ + a₂v₂ + \ldots + aₙvₙ, \]
where aᵢ ∈ C, (1 ≤ i ≤ n) is called a linear combination of vᵢ (1 ≤ i ≤ n).

Definition 8. The vectors v₁, v₂, \ldots, vₙ are said to span or generate V or are said to form a spanning set of V if V = span{v₁, v₂, \ldots, vₙ}. Alternatively, vᵢ ∈ V (1 ≤ i ≤ n) span V, if for every vector v ∈ V there exist scalars aᵢ ∈ C (1 ≤ i ≤ n) such that
\[ v = a₁v₁ + a₂v₂ + \ldots + aₙvₙ, \]
i.e., v is a linear combination of
\[ a₁v₁ + a₂v₂ + \ldots + aₙvₙ. \]

Remark 9. If V = span{v₁, v₂, \ldots, vₙ}, then each vector v ∈ V can be written as a linear combination of the vectors v₁, v₂, \ldots, vₙ. Spanning sets have the property that each vector in V has exactly one representation as a linear combination of these vectors.

Definition 10. Let V be a vector space over a field C. The vectors vᵢ ∈ V (1 ≤ i ≤ n) are said to be linearly independent over C, or simply independent, if it satisfies the following condition: if
\[ s₁v₁ + s₂v₂ + \ldots + sₙvₙ = 0, \]
then
\[ s₁ = s₂ = \ldots = sₙ = 0. \]

Otherwise, the vectors that are not linearly independent, are said to be linearly dependent, or simply dependent.

Remark 11. The trivial linear combination of the vectors vᵢ, (1 ≤ i ≤ n) is the one with every coefficient zero
\[ 0v₁ + 0v₂ + \ldots + 0vₙ. \]

Definition 12. A set of vectors {e₁, e₂, \ldots, eₙ} is called a basis of V if it satisfies the following two conditions
1° e₁, e₂, \ldots, eₙ are linearly independent,
2° V = span{e₁, e₂, \ldots, eₙ}.

Definition 13. A vector space V is said to be of finite dimension n or to be n-dimensional, written dim V = n, if V contains a basis with n elements.

Definition 14. The vector space {0} is defined to have dimension 0.

3 PARADOX APPEARANCE

Chemistry as other natural sciences is not immune of paradoxes. Unlike other natural sciences, in chemistry
Not yet! For instance, one may show the powerlessness of that procedure by the following two counterexamples.

**Example 1.**

\[
2993 \text{ C}_{2052} \text{H}_{4604} \text{N}_{516} \text{O}_{232} \text{S}_{8} \text{Fe}_{4} + 9300568 \text{ CsClO}_{4} + 408724 \text{ Li}_{2} \text{PO}_{4} \rightarrow 11972 \text{ Li}_{2} \text{Fe} \text{CN}_{6} + 8763504 \text{ CO}_{2} + 2370456 \text{ CsNO}_{3} + 1178284 \text{ LiHCl}_{2} + 408724 \text{ H}_{3} \text{PO}_{4} + 6944000 \text{ CsO}_{0.99} \text{Cl} + 23944 \text{ H}_{2} \text{S} + 5753504 \text{ H}_{2} \text{O}.
\]

What is the valence of C, N, S and Fe?

**Example 2.**

\[
249000 \text{ C}_{738} \text{H}_{1166} \text{N}_{812} \text{O}_{203} \text{S}_{2} \text{Fe} + 109813376 \text{ Pb(NO)}_{3} + 46812000 \text{ HSiCl}_{3} + 23148534 \text{ CoCl}_{2} + 7716178 \text{ Co}_{3}(\text{PO}_{4})_{2} + 124500 \text{ H}_{2} \text{SO}_{4} \rightarrow 2370456 \text{ CsNO}_{3} + 1178284 \text{ LiHCl}_{2} + 28657310960 \text{ H}_{3} \text{PO}_{4} + 11972 \text{ Li}_{2} \text{Fe} \text{CN}_{6} + 408724 \text{ H}_{3} \text{PO}_{4} + 9300568 \text{ CsClO}_{4} + 13945051000 \text{ Tl}_{1.989}(\text{SO}_{3})_{3} + 13889187000 \text{ Pr}_{1.997}(\text{SO}_{4})_{3} + 15313500 \text{ C}_{12} \text{H}_{22} \text{O}_{11} + 7716178 \text{ P}_{2} \text{O}_{5} + 109813376 \text{ Pb(NO)}_{3}.\]

Not yet! For instance, one may show the powerlessness of that procedure by the following two counterexamples.

**Example 1.**

\[
\begin{align*}
 x \text{ NaNO}_{2} + x \text{ FeSO}_{4} + x \text{ H}_{2} \text{SO}_{4} & \rightarrow x \text{ NaHSO}_{3} + x \text{ Fe}_{2}(\text{SO}_{4})_{2} + x \text{ NO}. \\
\end{align*}
\]

In the above chemical reaction N and Fe change the valence, but chemical equation is impossible! Thus, is by the procedure of formal balance numbers useless of that procedure by the following two counterexamples? Obviously, the answer is negative! It is merely a picture of the old chemical traditionalism. Or, maybe it is a lonely case lost in the newest progressive and contemporary mathematical generalism!

**Example 2.**

\[
\begin{align*}
 x_{1} \text{ NaNO}_{2} + x_{2} \text{ FeSO}_{4} + x_{3} \text{ H}_{2} \text{SO}_{4} & \rightarrow x_{4} \text{ NaHNO}_{3} + x_{5} \text{ Fe}_{2}(\text{SO}_{4})_{2} + x_{6} \text{ NO}. \\
\end{align*}
\]

Not yet! For instance, one may show the powerlessness of that procedure by the following two counterexamples.

**Example 3.**

\[
\begin{align*}
 x_{1} \text{ NaNO}_{2} + x_{2} \text{ FeSO}_{4} + x_{3} \text{ H}_{2} \text{SO}_{4} & \rightarrow x_{4} \text{ NaHNO}_{3} + x_{5} \text{ Fe}_{2}(\text{SO}_{4})_{2} + x_{6} \text{ NO}. \\
\end{align*}
\]

The same previous question holds for this counterexample too. This means, the solution of the above equations is based on very sophisticated matrix methods\textsuperscript{21-23}, but in no case on the change in oxidation number procedure!

As consequence of that, same holds for so-called *formal balance numbers*. That procedure is useless to balance complex chemical equations as it does not give effects for balancing chemical equations. Still, there are many causes for arguing why that elementary procedure is useless, but the above mentioned two counterexamples are enough to show that it is ineffective.

2° Consider this simple reaction

**Example 4.**

\[
\begin{align*}
 x_{1} \text{ Fe}_{2}(\text{SO}_{4})_{3} + x_{2} \text{ NO} + x_{3} \text{ CO}_{2} & \rightarrow x_{4} \text{ Fe}_{2}(\text{SO}_{4})_{3} + x_{5} \text{ CO}_{2} + x_{6} \text{ NO}. \\
\end{align*}
\]

Not yet! Since it is not possible to determine the valence of Fe, Tl and Pr. Then, on what basis the author\textsuperscript{1} states, that his procedure (there called method) is proposed for fast and easy balancing of complex redox equations? On top of all, he states that the procedure (there called method) is probably the fastest of all possible methods! Really, a very modest statement is offered by the author, which is wrong, not only because of today’s current balancing methods view point, but also from an earlier view point, when that procedure was published.

Is it a *method* when somebody can find on every step counterexamples? Obviously, the answer is negative! It is merely a picture of the old chemical traditionalism. Or, maybe it is a lonely case lost in the newest progressive and contemporary mathematical generalism!

**4 PARADOX RESOLUTION BY A NEW COMPLEX VECTOR METHOD**

With the purpose of solution of the paradox, in this section a new complex vector method of balancing chemical equations will be developed. This method is founded on the theory of n-dimensional complex vector spaces.

**Theorem 15.** Suppose that chemical equation

\[
x_{i}v_{1} + x_{i}v_{2} + \ldots + x_{i}v_{n} = 0 ,
\]

where \( v_{i} (1 \leq i \leq n) \) are the molecules and \( x_{i} (1 \leq i \leq n) \) are unknown coefficients is a vector space \( V \) over the field \( C \) spanned by the vectors of the molecules \( v_{i} (1 \leq i \leq n) \). If any set of \( m \) vectors of the molecules in \( V \) is linearly independent, then \( m \leq n \).

**Proof.** Let be \( V = \text{span}\{v_{1}, v_{2}, \ldots, v_{n}\} \). We must show that every set \( \{u_{1}, u_{2}, \ldots, u_{m}\} \) of vectors in \( V \) with \( m > n \) fails to be linearly independent. This is accomplished by showing that numbers \( x_{1}, x_{2}, \ldots, x_{m} \) can be found, not all zero, such that

\[
x_{1}u_{1} + x_{2}u_{2} + \ldots + x_{m}u_{m} = 0.
\]

Since \( V \) is spanned by the vectors \( v_{1}, v_{2}, \ldots, v_{n} \), each vector \( u_{i} \) can be expressed as a linear combination of \( v_{i} \)

\[
u_{i} = a_{1}v_{1} + a_{2}v_{2} + \ldots + a_{n}v_{n}.
\]

Substituting these expressions into the preceding equation gives

\[
0 = \sum_{i=1}^{m} x_{i} \left( \sum_{j=1}^{n} a_{j} v_{j} \right) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{j} x_{j} \right) v_{i}
\]

This is certainly the case if each coefficient of \( v_{i} \) is zero, i. e., if

\[
\sum_{j=1}^{n} a_{j} x_{j} = 0 , \ (1 \leq i \leq n).
\]

This is a system of \( n \) equations with \( m \) variables \( x_{1}, x_{2}, \ldots, x_{m} \), and because \( m > n \), it has a nontrivial solution.
This is what we wanted. Now we shall prove the following results.

**Theorem 16.** Let $U$ be a subset of a vector space $V$ of the chemical equation (1) over the field $C$. Then $U$ is a subspace of $V$ if and only if it satisfies the following conditions

1. $0 \in U$, where $0$ is the zero vector of $V$.
2. If $u_1, u_2 \in U$, then $(u_1 + u_2) \in U$.
3. If $u \in U$, then $au \in U$, for all $a \in C$.

**Proof.** If $U$ is a subspace of $V$ of the chemical equation (1), it is clear by axioms (A1) and (S1) that the sum of two vectors in $U$ is again in $U$ and that any scalar multiple of a vector in $U$ is again in $U$. In other words, $U$ is closed under the vector addition and scalar multiplication of $V$. The converse is also true, i.e., if $U$ is closed under these operations, then all the other axioms are automatically satisfied. For instance, axiom (A3) asserts that holds $u_1 + u_2 = u_2 + u_1$, $\forall u_1, u_2 \in U$. This is clear because the equation is already true in $V$, and $U$ uses the same addition as $V$. Similarly, axioms (A1), (S2), (S3), (S5) and (S6) hold automatically in $U$, because they are true in $V$. All that remains is to verify axioms (A2) and (A5).

If (2), (3) and (4) hold, then axiom (A2) follows from (2) and axiom (A5) follows from (4), because $-u = (-1)u$ lies in $U$, $\forall u \in U$. Hence, $U$ is a subspace by the above discussion. Conversely, if $U$ is a subspace, it is closed under addition and scalar multiplication and this gives (3) and (4). If $z$ denotes the zero vector of $U$, then $z = 0z$ in $U$. But, $0z = 0$ in $V$, so $0 = z$ lies in $U$. This proves (2).

**Remark 17.** If $U$ is a subspace of $V$ of the chemical equation (1) over the field $C$, then $\{0\}$ and $V$ are subspaces of $V$.

**Proof.** $U = V$ clearly satisfies the conditions of the Theorem 16. As to $U = \{0\}$, it satisfies the conditions because $0 + 0 = 0$ and $a0 = 0$, $\forall a \in C$.

**Remark 19.** The vector space $\{0\}$ is called the zero subspace of $V$ of the chemical equation (1) over the field $C$. Since all zero subspaces look alike, we speak of the zero vector space and denote it by $0$. It is the unique vector space containing just one vector.

**Proposition 20.** If $v$ is a vector of some molecule in a vector space $V$ of the chemical equation (1) over the field $C$, then the set $Cv = \{av, v \in C \}$ of all scalar multiples of $v$ is a subspace of $V$.

**Proof.** Since $0 = 0v$, it is clear that $0$ lies in $Cv$. Given two vectors $av$ and $bv$ in $Cv$, their sum $av + bv = (a + b)v$ is also a scalar multiple of $v$ and so lies in $Cv$. Therefore $Cv$ is closed under addition. Finally, given $av$, $rav = (ra)v$ lies in $Cv$, so $Cv$ is closed under scalar multiplication. Now, if we take into account the Theorem 16, immediately follows the statement of the proposition.

**Theorem 21.** Let $U = \text{span}\{v_1, v_2, \ldots, v_n\}$ in a vector space $V$ of the chemical equation (1) over the field $C$. Then,

1. $U$ is a subspace of $V$ containing each of $v_i$ ($1 \leq i \leq n$).
2. $U$ is the smallest subspace containing these vectors in the sense that any subspace of $V$ that contains each of $v_i$ ($1 \leq i \leq n$), must contain $U$.

**Proof.** First we shall prove (5). Clearly,

$$0 = 0v_1 + 0v_2 + \ldots + 0v_n$$

belongs to $U$. If

$$v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$$

and

$$w = b_1v_1 + b_2v_2 + \ldots + b_nv_n$$

are two members of $U$ and $a \in U$, then

$$v + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \ldots + (a_n + b_n)v_n$$

and

$$av = (aa_1)v_1 + (aa_2)v_2 + \ldots + (aa_n)v_n$$

so both $v + w$ and $av$ lie in $U$. Hence, $U$ is a subspace of $V$. It contains each of $v_i$ ($1 \leq i \leq n$). For instance,

$$v = 0v_1 + 1v_2 + 0v_3 + \ldots + 0v_n$$

This proves (5).

Now, we shall prove (6). Let $W$ be subspace of $V$ that contains each of $v_i$ ($1 \leq i \leq n$). Since $W$ is closed under scalar multiplication, each of $av_i$ ($1 \leq i \leq n$) lies in $W$ for any choice of $a_i$ ($1 \leq i \leq n$) in $C$. But, then $av_i$ ($1 \leq i \leq n$) lies in $W$, because $W$ is closed under addition. This means that $W$ contains every member of $U$, which proves (6).

**Theorem 22.** The intersection of any number of subspaces of a vector space $V$ of the chemical equation (1) over the field $C$ is a subspace of $V$.

**Proof.** Let $\{W_i : i \in I\}$ be a collection of subspaces of $V$ and let $W = \bigcap (W_i : i \in I)$. Since each $W_i$ is a subspace, then $0 \in W_i$, $\forall i \in I$. Thus $0 \in W$. Assume $u, v \in W$. Then, $u, v \in W_i$, $\forall i \in I$. Since each $W_i$ is a subspace, then $(au + bv) \in W_i$, $\forall i \in I$. Therefore $(au + bv) \in W$. Thus $W$ is a subspace of $V$ of the chemical equation (1).

**Theorem 23.** The union $W_1 \cup W_2$ of subspaces of a vector space $V$ of the chemical equation (1) over the field $C$ need not be a subspace of $V$.

**Proof.** Let $V = C^2$ and let $W_1 = \{(a, 0) : a \in C\}$ and $W_2 = \{(0, b) : b \in C\}$. That is, $W_1$ is the $x$-axis and $W_2$ is the $y$-axis in $C^2$. Then $W_1$ and $W_2$ are subspaces of $V$ of the chemical equation (1). Let $u = (1, 0)$ and $v = (0, 1)$. Then the vectors $u$ and $v$ both belong to the union $W_1 \cup W_2$, but $u + v = (1, 1)$ does not belong to $W_1 \cup W_2$. Thus $W_1 \cup W_2$ is not a subspace of $V$.

**Theorem 24.** The homogeneous system of linear equations obtained from the chemical equation (1), in $n$
unknowns $x_1, x_2, \ldots, x_n$ over the field $\mathbb{C}$ has a solution set $W$, which is a subspace of the vector space $\mathbb{C}^n$.

**Proof.** The system is equivalent to the matrix equation $Ax = 0$. Since $A0 = 0$, the zero vector $\mathbf{0} \in W$. Assume $u$ and $v$ are vectors in $W$; i.e., $u$ and $v$ are solutions of the matrix equation $Ax = 0$. Then $Au = 0$ and $Av = 0$. Therefore, $u + v$, $b \in \mathbb{C}$, we have $A(u + bv) = au + bv = a0 + b0 = 0 + 0 = 0$. Hence $au + bv$ is a solution of the matrix equation $Ax = 0$, i.e., $au + bv \in W$. Thus $W$ is a subspace of $\mathbb{C}^n$.

**Theorem 25.** If $S$ is a subset of the vector space $V$ of the chemical equation (1) over the field $\mathbb{C}$, then

1. the set $\text{span}\{S\}$ is a subspace of $V$ of the chemical equation (1) over the field $\mathbb{C}$ which contains $S$.

2. $\text{span}\{S\} \subseteq W$, if $W$ is any subspace of $V$ of the chemical equation (1) over the field $\mathbb{C}$ containing $S$.

**Proof.** 1. If $S = \emptyset$, then $\text{span}\{S\} = \{0\}$, which is a subspace of $V$ containing the empty set $\emptyset$. Now assume $S \neq \emptyset$. If $v \in S$, then $v = v \in \text{span}\{S\}$, therefore $S$ is a subset of $\text{span}\{S\}$. Also, $\text{span}\{S\} \neq \emptyset$ because $S \neq \emptyset$. Now assume $v, w \in \text{span}\{S\}$; say

$$ v = a_1v_1 + \ldots + a_nv_n $$

and

$$ w = b_1w_1 + \ldots + b_kw_k, $$

where $v_i, w_j \in S$ and $a_i, b_j$ are scalars.

Then

$$ v + w = a_1v_1 + \ldots + a_nv_n + b_1w_1 + \ldots + b_kw_k, $$

and for any scalar $k$,

$$ kv = k(a_1v_1 + \ldots + a_nv_n) = kav_1 + \ldots + kav_n \in \text{span}\{S\} $$

since each is a linear combination of vectors in $S$. Thus $\text{span}\{S\}$ is a subspace of $V$ of the chemical equation (1) over the field $\mathbb{C}$ which contains $S$.

2. If $S = \emptyset$, then any subspace $W$ contains $S$, and $\text{span}\{S\} = \{0\}$ is contained in $W$. Now assume $S \neq \emptyset$ and assume $v_i, v \in S \subset V (1 \leq i \leq m)$. Then all multiples $av_i, W (1 \leq i \leq m)$ where $a_i \in \mathbb{C}$, and therefore the sum $a_1v_1 + \ldots + a_nv_n \in \text{span}\{S\}$ contains all linear combinations of elements of $S$. Consequently, $\text{span}\{S\} \subseteq W$, as claimed.

**Proposition 26.** If $W$ is a subspace of $V$ of the chemical equation (1) over the field $\mathbb{C}$, then $\text{span}\{W\} = W$.

**Proof.** Since $W$ is a subspace of $V$ of the chemical equation (1) over the field $\mathbb{C}$, $W$ is closed under linear combinations. Hence, $\text{span}\{W\} \subseteq W$. But $W \subseteq \text{span}\{W\}$. Both inclusions yield $\text{span}\{W\} = W$.

**Proposition 27.** If $S$ is a subspace of $V$ of the chemical equation (1) over the field $\mathbb{C}$, then $\text{span}\{\text{span}\{S\}\} = \text{span}\{S\}$.

**Proof.** Since $\text{span}\{S\}$ is a subspace of $V$, the above Proposition 26 implies that $\text{span}\{\text{span}\{S\}\} = \text{span}\{S\}$.

**Proposition 28.** If $S$ and $T$ are subsets of a vector space $V$ of the chemical equation (1) over the field $\mathbb{C}$, such that $S \subseteq T$, then $\text{span}\{S\} \subseteq \text{span}\{T\}$.

**Proof.** Assume $v \in \text{span}\{S\}$. Then

$$ v = a_1u_1 + \ldots + a_nu_n $$

where $a_i \in \mathbb{C}, (1 \leq i \leq r)$ and $u_i \in S (1 \leq i \leq r)$. But $S \subseteq T$; therefore every $u_i \in T (1 \leq i \leq r)$. Thus $v \in \text{span}\{T\}$. Accordingly, $\text{span}\{S\} \subseteq \text{span}\{T\}$.

**Proposition 29.** The $\text{span}\{S\}$ is the intersection of all the subspaces of a vector space $V$ of the chemical equation (1) over the field $\mathbb{C}$ which contains $S$.

**Proof.** Let $\{W\}$ be the collection of all subspaces of a vector space $V$ of the chemical equation (1) containing $S$, and let $W = \bigcap W$. Since each $W_i$ is a subspace of $V$, the set $W$ is a subspace of $V$. Also, since each $W_i$ contains $S$, the intersection $W$ contains $S$. Hence $\text{span}\{S\} \subseteq W$. On the other hand, span $\{S\}$ is a subspace of $V$ containing $S$. So $\text{span}\{S\} = W_k$ for some $k$. Then $W \subseteq W_k = \text{span}\{S\}$. Both inclusions give $\text{span}\{S\} = W$.

**Proposition 30.** If $\text{span}\{S\} = \text{span}\{S \cup \{0\}\}$, then one may delete the zero vector from any spanning set.

**Proof.** By Proposition 28, $\text{span}\{S\} \subseteq \text{span}\{S \cup \{0\}\}$. Assume $v \in \text{span}\{S \cup \{0\}\}$, say

$$ v = a_1u_1 + \ldots + a_nu_n + b_0 $$

where $a_i, b \in \mathbb{C}$, $1 \leq i \leq n)$ and $u_i \in S (1 \leq i \leq n)$.

Then $v = a_1u_1 + \ldots + a_nu_n$, and so $v \in \text{span}\{S\}$. Thus $\text{span}\{S \cup \{0\}\} \subseteq \text{span}\{S\}$. Both inclusions give $\text{span}\{S\} = \text{span}\{S \cup \{0\}\}$.

**Proposition 31.** If the vectors $v_i \in V$ $(1 \leq i \leq n)$ span a vector space $V$ of the chemical equation (1) over the field $\mathbb{C}$, then for any vector $w \in V$, the vectors $v_i, (1 \leq i \leq n)$ span $V$.

**Proof.** Let $v \in V$. Since the $v_i (1 \leq i \leq n)$ span $V$, there exist scalars $a_i (1 \leq i \leq n)$ such that $v = a_1v_1 + \ldots + a_nv_n + 0w$. Thus $w_i, (1 \leq i \leq n)$ span $V$.

**Proposition 32.** If $v_i (1 \leq i \leq n)$ span a vector space $V$ of the chemical equation (1) over the field $\mathbb{C}$, and for $k > 1$, the vector $v_k$ is a linear combination of the preceding vectors $v_i (1 \leq i \leq k – 1)$, then $v_i$, without $v_k$ span $V$; i.e., $\text{span}\{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\} = V$.

**Proof.** Let $v \in V$. Since the $v_i (1 \leq i \leq n)$ span $V$, there exist scalars $a_i \in \mathbb{C}, (1 \leq i \leq n)$ such that

$$ v = a_1v_1 + \ldots + a_nv_n. $$

Since $v_k$ is a linear combination of $v_i$ $(1 \leq i \leq k – 1)$, there exist scalars $b_i \in \mathbb{C}, (1 \leq i \leq k – 1)$ such that

$$ v_i = b_1v_1 + \ldots + b_kv_{k-1}. $$

Thus

$$ v = a_1v_1 + \ldots + a_nv_n + a_1b_1v_1 + \ldots + a_nb_nv_{k-1} + \ldots + a_nv_n $$

$$ = (a_1 + a_1b_1)v_1 + \ldots + (a_{k-1} + a_kb_{k-1})v_{k-1} + a_kv_n $$

$$ + a_{k+1}v_{k+1} + \ldots + a_nv_n. $$

Therefore,

$$ \text{span}\{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\} = V. $$

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Proposition 33. If \( W_i \) (\( 1 \leq i \leq k \)) are subspaces of a vector space \( V \) of the chemical equation (1) over the field \( \mathbb{C} \), for which \( W_1 \subseteq W_2 \subseteq \ldots \subseteq W_k \) and
\[
W = W_1 \cup W_2 \cup \ldots \cup W_k,
\]
then \( W \) is a subspace of \( V \).

Proof. The zero vector \( \mathbf{0} \in W_1 \), hence \( \mathbf{0} \in W \).
Assume \( u, v \in W \). Then, there exist \( j_1 \) and \( j_2 \) such that \( u \in W_{j_1} \) and \( v \in W_{j_2} \). Let \( j = \max(j_1, j_2) \). Then \( W_j \subseteq W_{j_1} \subseteq W \) and so \( u, v \in W_j \). But \( W_j \) is a subspace. Therefore \( (u + v) \in W_j \) and for any scalar \( s \) the multiple \( su \in W \). Since \( W_j \subseteq W \), we have \( (u + v), su \in W \). Thus \( W \) is a subspace of \( V \).

Proposition 34. If \( W_i \) (\( 1 \leq i \leq k \)) are subspaces of a vector space \( V \) of the chemical equation (1) over the field \( \mathbb{C} \) and \( S \) (\( 1 \leq i \leq k \)) span \( W_i \) (\( 1 \leq i \leq k \)), then \( S \subseteq S_j \) is a subspace.

Proof. Let \( v \in W_i \). Then there exists \( j \) such that \( v \in W_j \). Then \( v \in \text{span}\{S\} \subseteq \text{span}\{S_j\} \). Therefore \( W_j \subseteq \text{span}\{S\} \). But \( S_j \subseteq W_j \) is a subspace. Hence \( \text{span}\{S\} \subseteq W \). Both inclusions give \( \text{span}\{S\} = W \), i.e., \( S \) spans \( W \).

Theorem 35. Let \( \{v_1, v_2, \ldots, v_n\} \) be a linearly independent set of vectors in a vector space \( V \) of the chemical equation (1) over the field \( \mathbb{C} \), then the following conditions
\[
1^\circ \{v_1, v_2, \ldots, v_n\} \text{ is a linearly independent set},
\]
\[
2^\circ v \text{ does not lie in } \{v_1, v_2, \ldots, v_n\},
\]
are equivalent for a vector \( v \) in \( V \).

Proof. Assume \( 1^\circ \) is true and assume, if possible, that \( v \) lies in \( \text{span}\{v_1, v_2, \ldots, v_n\} \), say,
\[
v = a_1v_1 + \ldots + a_nv_n.
\]
Then
\[
v - a_1v_1 - \ldots - a_nv_n = 0
\]
is a nontrivial linear combination, contrary to \( 1^\circ \). So \( 1^\circ \) implies \( 2^\circ \). Conversely, assume that \( 2^\circ \) holds and assume that
\[
av + a_1v_1 + \ldots + a_nv_n = 0.
\]
If \( a \neq 0 \), then
\[
v = (-a/a)v_1 + \ldots + (-a/a)v_n,
\]
contrary to \( 2^\circ \). So \( a = 0 \) and
\[
av_1 + \ldots + a_nv_n = 0.
\]
This implies that
\[
a_1 = \ldots = a_n = 0,
\]
because the set \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent. By this is proved that \( 2^\circ \) implies \( 1^\circ \).

Proposition 36. Let \( \{v_1, v_2, \ldots, v_n\} \) be linearly independent in a vector space \( V \) of the chemical equation (1) over the field \( \mathbb{C} \), then \( \{a_1v_1, a_2v_2, \ldots, a_nv_n\} \), such that the numbers \( a_i \) (\( 1 \leq i \leq n \)) are all nonzero, is also linearly independent.

Proof. Suppose a linear combination of the new set vanishes
\[
s_1(a_1v_1) + s_2(a_2v_2) + \ldots + s_n(a_nv_n) = \mathbf{0},
\]
where \( s_i \) (\( 1 \leq i \leq n \)) lie in \( \mathbb{C} \).

Then
\[
s_1a_1 = s_2a_2 = \ldots = s_na_n = 0
\]
by the linear independence of \( \{v_1, v_2, \ldots, v_n\} \). The fact that each \( a_i \neq 0 \) (\( 1 \leq i \leq n \)) now implies that \( s_1 = s_2 = \ldots = s_n = 0 \).

Proposition 37. No linearly independent set of vectors of molecules can contain the zero vector.

Proof. The set \( \{\mathbf{0}, v_1, v_2, \ldots, v_n\} \) cannot be linearly independent, because
\[
10 + 0v_1 + \ldots + 0v_n = \mathbf{0},
\]
is a nontrivial linear combination that vanishes.

Theorem 38. Let \( \{v_1, v_2, \ldots, v_n\} \) be a linearly independent set of vectors of molecules in a vector space \( V \) of the chemical equation (1) over the field \( \mathbb{C} \), if and only if \( v_i \) is a linear combination of the others.

Proof. Assume that \( \{v_1, v_2, \ldots, v_n\} \) is linearly dependent. Then, some nontrivial linear combination vanishes, i.e.,
\[
ah_1v_1 + ah_2v_2 + \ldots + h_nv_n = \mathbf{0},
\]
where some coefficient is not zero. Suppose \( a_1 \neq 0 \). Then
\[
v_1 = (-a_1/a_1)v_1 + \ldots + (-a_1/a_1)v_n,
\]
gives \( v_1 \) as a linear combination of the others. In general, if \( a_i \neq 0 \), then a similar argument expresses \( v_i \) as linear combination of the others.

Conversely, suppose one of the vectors is a linear combination of the others, i.e.,
\[
v_1 = ah_2v_2 + \ldots + h_nv_n,
\]
Then, the nontrivial linear combination \( 1v_1 - ah_2v_2 - \ldots - h_nv_n \) equals zero, so the set \( \{v_1, v_2, \ldots, v_n\} \) is not linearly independent, i.e., it is linearly dependent. A similar argument works if any \( v_i \) (\( 1 \leq i \leq n \)) is a linear combination of the others.

Theorem 39. Let \( V \neq \mathbf{0} \) be a vector space of the chemical equation (1) over the field \( \mathbb{C} \), then
\[
1^\circ \text{each set of linearly independent vectors is a part of a basis of } V,
\]
\[
2^\circ \text{each spanning set } V \text{ contains a basis of } V.
\]

Proof. \( 1^\circ \) Really. If \( V \) is a vector space that is spanned by a finite number of vectors, we claim that \( \text{any linearly independent subset } S = \{v_1, v_2, \ldots, v_k\} \) of \( V \) is contained in a basis of \( V \). This is certainly true if \( V = \text{span}\{S\} \) because then \( S \) is itself a basis of \( V \). Otherwise, choose \( v_{k+1} \) outside \( \text{span}\{S\} \). Then \( S = \{v_1, v_2, \ldots, v_k, v_{k+1}\} \) is linearly independent by Theorem 35. If \( V = \text{span}\{S\} \), then \( S \) is the desired basis containing \( S \). If not, chose \( v_{k+2} \) outside \( \text{span}\{v_1, v_2, \ldots, v_k, v_{k+1}\} \) so that \( S_2 = \{v_1, v_2, \ldots, v_{k+2}\} \) is a basis of \( V \).
... $v_2, v_3, v_{121}, v_{122}$] is linearly independent. Continue this
process. Either a basis is reached at some stage or, if not,
arity large independent sets are found in $V$ But this
later possibility cannot occur by the Theorem 15 because
$V$ is spanned by a finite number of vectors.

$2^\circ$ Let $V = \text{span}\{v_1, v_2, \ldots, v_n\}$, where (as $V \neq \emptyset$) we
may assume that each $v_i \neq 0$. If $\{v_1, v_2, \ldots, v_n\}$ is linearly
independent, it is itself a basis and we are finished. If not,
then according to the Theorem 38, one of these vectors
lies in the span of the others. Relabeling if
ecessary, it is assumed that $v_1$ lies in $\text{span}\{v_2, \ldots, v_n\}$ so
that $V = \text{span}\{v_2, \ldots, v_n\}$. Now repeat the argument. If the
set $\{v_2, \ldots, v_n\}$ is linearly independent, we are
finished. If not, we have (after possible relabeling) $V = \text{span}\{v_1, \ldots, v_m\}$. Continue this process and if a basis is
encountered at some stage, we are finished. If not, we
ultimately reach $V = \text{span}\{v_m\}$. But then $\{v_m\}$ is a basis
because $v_m \neq 0$ ($V \neq \emptyset$).

$3^\circ$ $V$ has a spanning set of $n$ vectors, one of which is
nonzero because $V \neq \emptyset$. Hence $3^\circ$ follows from $2^\circ$.

Corollary 40. A nonzero vector space $V$ of the
chemical equation (1) over the field $C$ is finite dimen-
sional only if it can be spanned by finitely many vectors.

Theorem 41. Let $V$ be a vector space of the chemical
equation (1) over the field $C$ and $\dim V = n > 0$, then
$1^\circ$ no set of more than $n$ vectors in $V$ can be linearly
independent,
$2^\circ$ no set of fewer than $n$ vectors can span $V$.

Proof. $V$ can be spanned by $n$ vectors (any basis) so
$1^\circ$ restates the Theorem 15. But the $n$ basis vectors are
also linearly independent, so no spanning set can have
fewer than $n$ vectors, again by Theorem 15. This gives
$2^\circ$.

Theorem 42. Let $V$ be a vector space of the chemical
equation (1) over the field $C$ and $\dim V = n > 0$, then
$1^\circ$ any set of $n$ linearly independent vectors in $V$ is a
basis (that is, it necessarily spans $V$),
$2^\circ$ any spanning set of $n$ nonzero vectors in $V$ is a
basis (that is, they are necessarily linearly independent).

Proof. $1^\circ$ If the $n$ independent vectors do not span $V,$
they are a part of a basis of more than $n$ vectors by pro-
erty $1^\circ$ of the Theorem 39. This contradicts Theorem
41.
$2^\circ$ If the $n$ vectors in a spanning set are not linearly
independent, they contain a basis of fewer than $n$ vectors
by property $2^\circ$ of Theorem 39, contradicting Theorem
41.

Theorem 43. Let $V$ be a vector space of dimension $n$
of the chemical equation (1) over the field $C$ and let $U$ and
$W$ denote subspaces of $V,$ then
$1^\circ$ $U$ is finite dimensional and $\dim U \leq n$,
$2^\circ$ any basis of $U$ is a part of a basis for $V$,
$3^\circ$ if $U \subseteq W$ and $\dim U = \dim W$ then $U = W$.

Proof. $1^\circ$ If $U = 0$, then $\dim U = 0$. By Definition 14. So
assume $U \neq 0$ and choose $u_1 \neq 0$ in $U$. If $U = \text{span}\{u_1\}$,
then $\dim U = 1$. If $U \neq \text{span}\{u_1\}$, choose $u_2$ in $U$ outside
span $\{u_1\}$. Then $\{u_1, u_2\}$ is linearly independent by
Theorem 35. If $U = \text{span}\{u_1, u_2\}$, then $\dim U = 2$. If not,
repeat the process to find $u_3$ in $U$ such that $\{u_1, u_2, u_3\}$ is
linearly independent and continue in this way. The pro-
cess must terminate because the space $V$ (having dimen-
sion $n$) cannot contain more than $n$ independent vectors.
Therefore $U$ has a basis of at most $n$ vectors, proving $1^\circ$.

$2^\circ$ This follows from $1^\circ$ and Theorem 39.
$3^\circ$ Let $\dim U = \dim W = m$. Then any basis $\{u_1, u_2, \ldots,$
$u_m\}$ of $U$ is an independent set of $m$ vectors in $W$ and so
is a basis of $W$ by Theorem 42. In particular, $\{u_1, u_2, \ldots,$
$u_m\}$ spans $W$ so, because it also spans $U$, $W = \text{span}\{u_1,$
$u_2, \ldots, u_m\} = U$. By this is proved $3^\circ$.

Proposition 44. If $U$ and $W$ are subspaces of a vector
space $V$ of the chemical equation (1) over the field $C$,
then $U + W$ is a subspace of $V$.

Proof. Since $U$ and $W$ are subspaces, $0 \in U$ and $0 \in W$.
Hence $0 = 0 + 0 \in U + W$. Assume $u, v \in U + W$. Then there exist $u, u' \in U$ and $v, v' \in W$ such that $u = u + w$ and $v' = u' + w'$. Since $U$ and $W$ are subspaces, $u + u' \in U$ and $w + w' \in W$ and for any scalar $k$, $ku \in U$ and
and $kw \in W$. Accordingly, $v + v' = (u + w) + (u' + w') = (u + u') + (w + w') \in U + W$ and for any scalar $k$, $kv = ku$ and $kw \in U + W$. Thus $U + W$ is a subspace of $V$.

Proposition 45. If $U$ and $W$ are subspaces of a vector
space $V$ of the chemical equation (1) over the field $C$, then $U$ and $W$ are contained in $U + W$.

Proof. Let $u \in U$. By hypothesis $W$ is a subspace of $V$ and
so $0 \in U$. Hence $u = u + 0 \in U + W$. Accordingly,
$U$ is contained in $U + W$. Similarly, $W$ is contained in $U + W$. By this the proof is finished.

Proposition 46. If $U$ and $W$ are subspaces of a vector
space $V$ of the chemical equation (1) over the field $C$, then
$U + W$ is the smallest subspace of $V$ containing $U$ and
$W$, i.e., $U + W = \text{span}\{U, W\}$.

Proof. Since $U + W$ is a subspace of $V$ containing both
$U$ and $W$, it must also contain the linear span of $U$ and
$W$, i.e., $\text{span}\{U, W\} \subseteq U + W$.

On the other hand, if $v \in U + W$, then $v = u + w = 1u + 1w$, where $u \in U$ and $w \in W$. Hence, $v$ is a linear combination of elements in $U \cup W$ and so belongs to $\text{span}\{U, W\}$. Therefore, $U + W \subseteq \text{span}\{U, W\}$. Both
inclusions give us the required result.

Proposition 47. If $W$ is a subspace of a vector space
$V$ of the chemical equation (4. 2) over the field $C$, then $W + W = W$.

Proof. Since $W$ is a subspace of $V$, we have that $W$ is
closed under vector addition. Therefore, $W + W \subseteq W$. By
Proposition 45, $W \subseteq W + W$. Thus, $W + W = W$. By this the
proof is finished.

Proposition 48. If $U$ and $W$ are subspaces of a vector
space $V$ of the chemical equation (1) over the field $C$, such
that $U = \text{span}\{S\}$ and $W = \text{span}\{T\}$, then $U + W = \text{span}\{S \cup T\}$.

Proof. Since $S \subseteq U \subseteq U + W$ and $T \subseteq W \subseteq U + W$, we have $S \cup T \subseteq U + W$. Hence $\text{span}\{S \cup T\} \subseteq U + W$. 200,

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Now assume \( v \in U + W \). Then \( v = u + w \), where \( u \in U \) and \( w \in W \). Since \( U = \text{span}\{S\} \) and \( W = \text{span}\{T\} \), \( u = a_1 u_1 + \ldots + a_r u_r \) and \( w = b_1 w_1 + \ldots + b_s w_s \), where \( a_i, b_j \in C \), \( u_i \in S \), and \( w_i \in T \). Then \( v = u + w = a_1 u_1 + \ldots + a_r u_r + b_1 w_1 + \ldots + b_s w_s \). Thus, \( U + W \subseteq \text{span}\{S \cup T\} \) and both inclusions imply \( U + W = \text{span}\{S \cup T\} \).

**Proposition 49.** If \( U \) and \( W \) are subspaces of a vector space \( V \) of the chemical equation (1) over the field \( C \), then \( V = U + W \) if every \( v \in V \) can be written in the form \( v = u + w \), where \( u \in U \) and \( w \in W \).

**Proof.** Assume, for any \( v \in V \), we have \( v = u + w \) where \( u \in U \) and \( w \in W \). Then \( v \in U + W \) and so \( V \subseteq U + W \). Since \( U \) and \( V \) are subspaces of \( V \), we have \( U + W \subseteq V \). Both inclusions imply \( V = U + W \).

By this, in whole is given the skeleton of the complex vector method.

### 5 AN APPLICATION OF THE MAIN RESULTS

In this section the above complex vector method will be applied on some chemical equations for their balancing. All chemical equations balanced here appear for the first time in professional literature and they are chosen with an intention to avoid all well-known to date chemical equations which were repeated many times in the chemical journals for explanation of certain particular techniques for balancing of some chemical equations using only atoms with integer oxidation numbers.

1° First, we shall consider the case when the chemical reaction is infeasible.

**Example 5.** Consider this chemical reaction

\[
\begin{align*}
    z_1 KFe(CN)_6 & + z_2 K_2S_2O_3 \\
    \rightarrow z_3 CO_2 & + z_4 K_2SO_4 + z_5 NO_2 + z_6 FeS,
\end{align*}
\]

where \( z_i \in C \), \( (1 \leq i \leq 6) \).

From the following scheme

<table>
<thead>
<tr>
<th>( K\text{Fe(CN)}_6 )</th>
<th>( K\text{SO}_4 )</th>
<th>( CO_2 )</th>
<th>( K\text{SO}_4 )</th>
<th>( NO_2 )</th>
<th>( FeS )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>( v_2 )</td>
<td>( v_3 )</td>
<td>( v_4 )</td>
<td>( v_5 )</td>
<td>( v_6 )</td>
</tr>
<tr>
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<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Fe</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>N</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>O</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

follows this vector equation

\[
z_1 v_1 + z_2 v_2 = z_3 v_3 + z_4 v_4 + z_5 v_5 + z_6 v_6.
\]

i. e.,

\[
\begin{align*}
    z_1 (4, 1, 6, 0, 0, 0) & + z_2 (2, 0, 0, 0, 2, 3) \\
    = z_3 (0, 0, 1, 0, 0, 2) & + z_4 (2, 0, 0, 0, 1, 4) \\
    + z_5 (0, 0, 0, 1, 0, 0) & + z_6 (0, 1, 0, 0, 0, 0),
\end{align*}
\]

or

\[
(4z_1 + 2z_2, z_1, 6z_1, 6z_1, 2z_2, 3z_2)^T = (2z_1, 2z_1, 3z_1, z_1, 3z_1, 4z_1 + 2z_2)^T.
\]

From the system of linear equations

\[
\begin{align*}
    4z_1 + 2z_2 &= 2z_1 \\
    z_1 &= z_1 \\
    6z_1 &= z_1 \\
    6z_1 &= z_1 \\
    2z_2 &= 4z_1 + 2z_2, \\
    3z_2 &= 2z_1 + 4z_1 + 2z_2,
\end{align*}
\]

one obtains the contradictions \( z_2 = 3z_1 \) and \( z_2 = 4z_1/3 \), that means that the system is inconsistent. According to Definition 8, the vectors \( v_1, v_2, v_3, v_4, v_5 \) and \( v_6 \) of the molecules of the chemical reaction (5, 1) do not generate a vector space \( V \) and according to the Definition 10 they are linearly independent, i. e., we have only a trivial solution \( z_i = 0 \), \( (1 \leq i \leq 6) \), that means that the chemical reaction is infeasible.

2° Next, we shall consider the case when the chemical reaction is feasible and it has a unique solution.

This type of chemical equations really is the most appropriate for study the process of balancing chemical equations, because it gives an excellent opportunity for application of the group theory.

At once, we would like to emphasize here, that by application of groups theory one may determine Sylow subgroups, conjugacy classes of maximal subgroups, proper normal subgroups, and so on. The main reason why we confined ourselves to the next group of calculations is limitation of the size of the work.

**Example 6.** Consider this chemical reaction

\[
\begin{align*}
    z_1 Fe_2(SO_4)_3 & + z_2 PrTlTe_3 + z_3 H_2PO_4 \\
    \rightarrow z_4 Fe_{0.996}(H_2PO_4)_2 \cdot H_2O & + z_5 H_3PO_4 + z_6 Pr_{1.998}(SO_4)_3 \cdot H_2O \\
    + z_7 H_2O & + z_8 P_2O_7 + z_9 H_2S, \ z_i \in C \), \ (1 \leq i \leq 9) \).
\end{align*}
\]

According to the scheme given below

<table>
<thead>
<tr>
<th>( Fe_2(SO_4)_3 )</th>
<th>( PrTlTe_3 )</th>
<th>( H_2PO_4 )</th>
<th>( Fe_{0.996}(H_2PO_4)_2 \cdot H_2O )</th>
<th>( H_3PO_4 )</th>
<th>( Pr_{1.998}(SO_4)_3 \cdot H_2O )</th>
<th>( H_2O )</th>
<th>( P_2O_7 )</th>
<th>( H_2S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
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<td>( v_3 )</td>
<td>( v_4 )</td>
<td>( v_5 )</td>
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follows this vector equation
\[ z_i v_1 + z_i v_2 + z_i v_3 = z_i v_4 + z_i v_5 + z_i v_6, \]
\[ i.e., \]
\[ (z_i, 2z_i, 0, 0)^T + (0, 0, -z_i, 0)^T \]
\[ + (0, 0, 0, 2z_i)^T \]
\[ + (0, 0, 0, 2z_i)^T, \]
\[ i.e., \]
\[ (z_i, z_i, z_i, 0)^T \]
\[ = (z_i, z_i, z_i, 0)^T, \]
\[ or \]
\[ (z_i, z_i, z_i, 0)^T \]
\[ = (0, 0, 2z_i, -z_i, 0)^T, \]
\[ i.e., \]
\[ immediately \ follows \ this \ system \ of \ linear \ equations \]
\[ \begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4 \\
  z_5 \\
  z_6
\end{bmatrix} =
\begin{bmatrix}
  2z_1 \\
  2z_2 \\
  2z_3 \\
  2z_4 \\
  2z_5 \\
  2z_6 \\
  2z_7
\end{bmatrix}, \]
\[ which \ general \ solution \ is \]
\[ z_3 = 2z_1, \quad z_4 = z_1, \quad z_5 = -2z_1 + 2z_2, \quad z_6 = 2z_1 - z_2, \]
\[ where \ z_1 \ and \ z_2 \ are \ arbitrary \ complex \ numbers. \]

Now, balanced general chemical reaction has this form
\[ z_1 TiO_2 + z_2 C + z_3 Cl_2 \rightarrow z_4 TiCl_4 \]
\[ + (-2z_1 + 2z_2) CO + (2z_1 - z_2) CO_2, \]
\[ (\forall z_i, z_i \in C). \]

According to the Definition 8, the vectors \( v_i \), \( v_2 \), \( v_3 \), \( v_4 \), \( v_5 \), and \( v_6 \) of the molecules of the chemical reaction generate infinite number of vector spaces \( V_i \), and according to the Definition 10 they are linearly dependent, \( i.e. \), we have an infinite number of solutions \( (z_i, z_i, \ldots, z_i, z_i) \), \( (\forall z_i, z_i \in C) \), that means that the this chemical reaction is non-unique.

6 DISCUSSION

In his previous work \cite{22}, the author announced a decrease of barren intuitionism from of chemistry and its substitution by an elegant formalism from one side, and substitution of the old chemical traditionalism by a new mathematical generalism, from other side. This announcement is realized in this work that gives a new contribution to the theory as well as practice of balancing chemical equations.

The complex vector method of balancing chemical equations augmented the research field in chemistry and made obsolete the old traditional approach, and gave reliable results for paradox resolution.

By this work will begin consideration of paradoxes in chemistry as a serious object, and it will increase researchers’ carefulness to avoid appearance of paradoxes.

7 CONCLUSION

The new complex mathematical method of balancing chemical equations, which was used for the solution of a paradox is farewell to the chemical tradition, which still...
respects the composers of *general chemistry* textbooks, that affirm that chemistry is a science which studies the structure of substances, how they react when combined or in contact, and how they behave under different conditions. These subjects include a great part of the matter to which chemistry was applied.

In this work the *foundation of chemistry* is enriched by one more new topic, and a contribution to a new formalization of chemistry founded by virtue of a new complex vector method of balancing chemical equations is offered. This work opens doors for the next research in chemistry for diagnostic of paradoxes and their resolution. It will accelerate the newest mathematical research in chemistry and it will surmount the barriers hampering the development of chemistry.

This work is a critical survey that requires changes of chemical thinking. Hence, it must be distinguished from the uncritical penetration, in which chemistry itself is developed sometimes.

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